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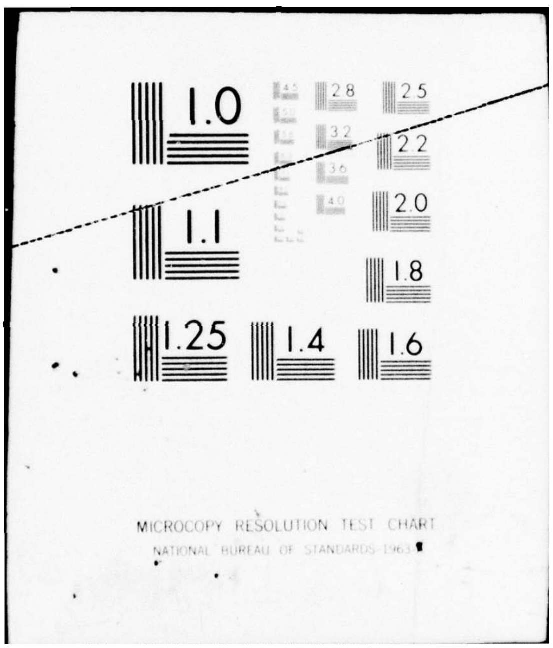
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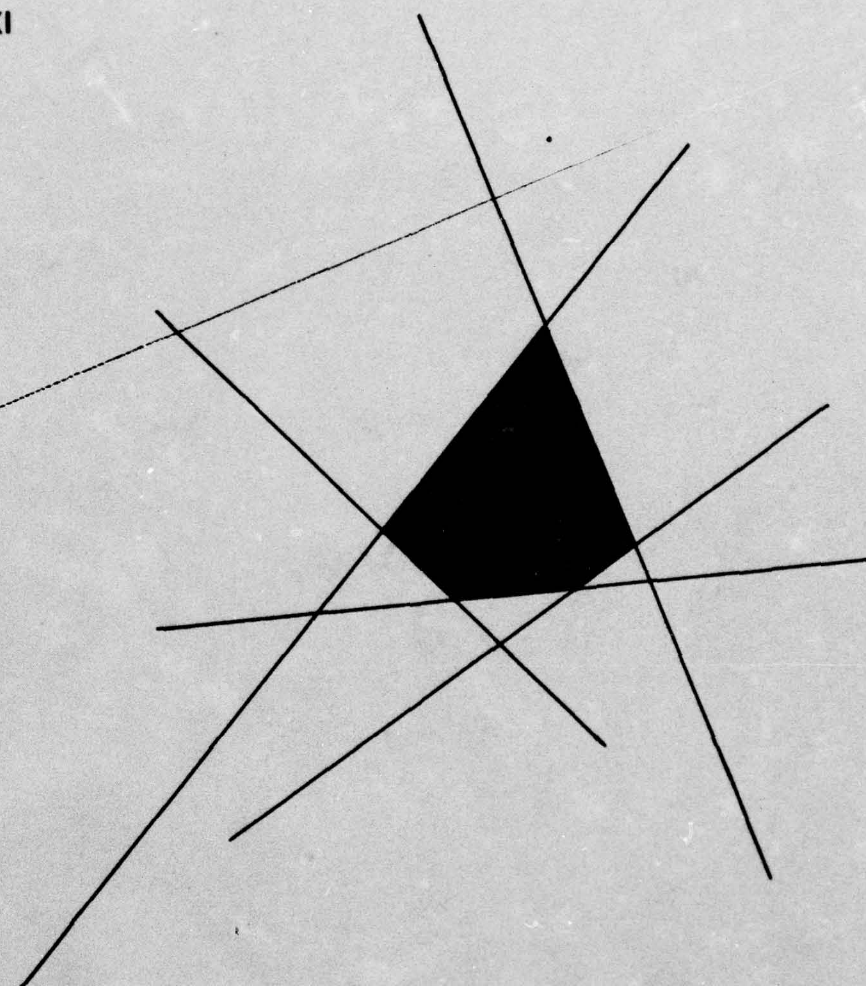
THEORY AND PRACTICE IN OPTIMAL REINSURANCE

by
JAN W. KWIATKOWSKI



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THEORY AND PRACTICE IN OPTIMAL REINSURANCE

Operations Research Center Research Report No. 76-21

Jan W. Kwiatkowski

July 1976

U. S. Army Research Office - Research Triangle Park

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ABSTRACT

In this paper the theory and practice of "optimality" in reinsurance, from the point of view of the primary insurer (cedent) are discussed.

The principal types of reinsurance contracts and their relative merits are described, and the question as to what constitutes an "optimal" contract is considered.

Some of the quantitative concepts relating to the various types of reinsurance are reviewed in the context of the practical considerations which may apply. The limitations and possible uses of these methods are illustrated.

NOTATION

R = Reserves

For a set of policies $\{i : i = 1, 2, \ldots, n\}$:

a_i = nominal sum insured

$$\begin{cases} x_i & = \text{observed value of random variable } \xi_i \text{, actual claim.} \\ X & = \text{the random variable } \sum_{i=1}^{n} \xi_i \text{, or its observed value } \sum_{i=1}^{n} x_i \\ F_i(\xi_i) & = \text{distribution function of } \xi_i \text{.} \end{cases}$$

$$\begin{cases} \mu_i & = \text{expected value of claim} \equiv \text{net 'fair' premium.} \\ M & = \text{total fair premiums} = \sum_{i=1}^{n} \mu_i \text{ (volume).} \end{cases}$$

σ_i^2 = variance

$$\begin{cases} \pi_i & = \text{gross premium (including risk and expense loadings)} \\ \Pi & = \sum_{i=1}^{n} \pi_i \text{ .} \end{cases}$$

$$\begin{cases} \eta_i & = \text{expense loading charged by primary insurer} \\ E & = \sum_{i=1}^{n} \eta_i \mu_i \text{ .} \end{cases}$$

$$\begin{cases} \lambda_i & = \text{risk loading charged by primary insurer.} \\ \theta_i & = \lambda_i \mu_i = \text{actual risk premium.} \\ \Lambda & = \sum_{i=1}^{n} \lambda_i \mu_i = \sum_{i=1}^{n} \theta_i \text{ .} \end{cases}$$

$$\Pi = \sum_{i=1}^{n} \mu_i (1 + \eta_i + \lambda_i) = M + E + \Lambda$$

S = premium reserve = $\Pi - E^{\dagger} \equiv M + \Lambda$

$$\begin{cases} z_i & = \text{observed value of } \zeta_i \text{, retained claim liability.} \\ Z & = \sum_{i=1}^{n} z_i \text{ .} \end{cases}$$

$$\begin{cases} w_i & = \text{observed value of } \omega_i \text{, ceded liability; } w_i = x_i - z_i \text{ .} \\ W & = \sum_{i=1}^{n} w_i \text{ .} \end{cases}$$

†Expenses are assumed to be exactly met by expense loading.

μ_i^* = expected value of ceded liability $\equiv$ net 'fair' reinsurance premium.

M^* $= \sum_{i=1}^{n} \mu_i^*$.

π_i^* = gross 'net' reinsurance premium (including risk loading less commissions)†

Π^* $= \sum_{i=1}^{n} \pi_i^*$.

λ_i^* = reinsurance risk loading.

θ_i^* $= \lambda_i^* \mu_i^*$ = actual reinsurance risk premium.

Λ^* $= \sum_{i=1}^{n} \lambda_i^* \mu_i^* = \sum_{i=1}^{n} \theta_i^*$

Π^* $= \sum_{i=1}^{n} \mu_i^* (1 + \lambda_i^*) = M^* + \Lambda^*$

$u(\cdot)$ = utility function for insurance company.

U = expected value of u for some contract.

1. OBJECTIVES OF AN INSURANCE COMPANY

The role of an insurance company is to spread the risks* from various contingencies among all its policy-holders. Each individual pays a "risk-loading" on top of the premium required to cover the expected value of his risk, so that the company, and hence the policy-holders themselves, are protected, to some extent, against an above-average total of claims. However, even a large insurance company may not be immune to fluctuations in the total claims, and further protection may be obtained through reinsurance. The insurance company can modify the *distribution* of its total risk by ceding some function of its liability to the reinsurer in return for which it pays a premium and/or accepts some additional (independent) risk.

In order to compare the merits of different courses of action, it will be necessary to define a preference ordering over the resulting distributions of eventual claims. Usually, the net liability of the insurance company will be the statistic of interest, and the preference ordering will be defined in terms of some expected value, or the probability of a certain event. It is therefore important to define the objectives of the company, and to choose a preference ordering consistent with these objectives.

The insurance company's objectives will depend upon what type of business it transacts. For example, a life insurance company will be more concerned with long term stability than a casualty insurance company, underwriting on a year-to-year basis. The casualty company will still be interested, however, in maintaining its reserve funds to enable it to continue operation. The issue may be further complicated by the requirements of share-holders, who will want the company to produce a positive surplus, for which they may be prepared to undergo a greater or lesser degree of

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*The risk is the *random variable* equal to the amount of a liability to an individual or insurance company.

risk to their investment. Thus it happens that some reinsurance strategies are expressed in terms of minimizing the probability of ruin, others in terms of the expected value of discounted dividends, and others in terms of minimizing the variance of the net liability subject to restrictions on its expected value.

There need not necessarily be any conflict between the different objectives of the company, and optimization of different criteria may lead to the same strategy, but this will not in general be the case.

Ideally it would be possible to define a utility function over possible values of the company's monetary status at the end of a given period. This would be constructed so as to take account of the relative importance of various factors. In fact, there must be some such scale of utility in the minds of the managers of the company, but whether they could give it numerical values, or whether any two managers would agree on these values, is a different matter. In any case, some attempt to define a utility function should lead to greater clarity of thought in decision-making.

If a utility function could be accurately defined, then a reinsurance strategy could, in theory, be chosen so as to maximize its expected value. The coefficient of risk aversion, as defined by Pratt [21] would in general be positive, i.e. the utility function would be concave, which means that the company would be prepared to trade off a certain amount of cash for a reduction in variance of the outcome; in fact comparisons of possible reductions in variance are frequently used as a basis for optimization when utility functions cannot be described.

2. MOTIVES FOR REINSURANCE

2.1 Legal Requirements

Insurance companies are required by law to place initial gross premiums immediately into unearned premium reserve. This means that initial expenses have to be met out of surplus, and this may cause financial strain, particularly to smaller companies. Under "treaty" reinsurance (described below), the reinsurance company is responsible for maintaining its share of the unearned premium reserve, and the commission received by the insurer from the reinsurer can be used immediately to meet expenses.

2.2 Entering or Leaving a New Geographical Area, or a New Field of Business

A company may take the role of reinsurer as an entrée into an area in which it has no experience; on the other hand, it may relieve itself of any financial obligations in some area by reinsuring the whole of the section of business from which it wishes to withdraw.

2.3 Spreading of Risk

The most common need for reinsurance arises out of insurance companies' aversion to risk (in utility theory terms, $u''(x)/u'(x) < 0$). In particular, companies will wish to guard their reserves. In fact their principal motives will be similar to those of the original policyholders in buying insurance, and similar quantitative considerations will apply.

We will be concerned only with this last motive.

3. DESCRIPTION OF COMMON TYPES OF REINSURANCE CONTRACT*

3.1 Treaty Reinsurance

Under this type of reinsurance contract, the insurance company (cedent) retains the liability for some function $Z(x_1, x_2, \ldots, x_n)$ of the claims $x_1, x_2, \ldots, x_n$, arising from its policies from a certain sector or sectors of its business, which it undertakes during the course of the treaty. The balance, $X - Z$, where $X = \sum_{i=1}^{n} x_i$, is met by the reinsurer.

The reinsurer receives a premium, which is usually calculated as a percentage of the total gross premiums collected by the cedent for the appropriate policies.

In the cases of "prorata quota" and "surplus" treaties, (see below), the reinsurance company pays a commission which is calculated to cover the expense loading included in its share of the premium, since the original expenses are, in fact, met by the cedent, and to return to the cedent a minor part of the corresponding risk loading. Thus, in theory, the *net* premium received by the reinsurer is equal to the appropriate share of the "fair" premium (expected value of the ceded liability) plus the greater share of the corresponding risk loading. The cedent may receive a small additional "overriding" commission to pay for a share of the expenses of arranging the treaty.

In the case of "excess" reinsurance, there is generally no commission payable, but the total reinsurance premium will be arranged with similar considerations in view as for the quota and surplus treaties.

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*The names used for describing some of these contracts are often interchanged.

3.1.1 Quota ("prorata quota") Reinsurance

A fixed proportion, $K < 1$, of all claims is retained by the cedent;

i.e. Total retention $Z = \sum_{i=1}^{n} Kx_n = KX$

Total ceded, $W = (1 - K)X$

Net reinsurance premium (less commission),

$$\Pi^* = (1 - K)M(1 + \lambda^*) \quad \text{and} \quad \lambda^* = h\lambda$$

where M = total "fair" premiums received by cedent

λ = cedent's original risk loading

h = constant < 1

i.e. $\Pi^* = (1 - K)(M + h\Lambda)$

where Λ = total risk loading received by cedent.

3.1.2 Surplus ("sliding-scale quota") Reinsurance

The proportion ceded is not fixed, but is related to a fixed monetary sum, B, the cedent's retention.

Then if a_i is the nominal sum insured on a certain policy, and if x_i is the actual claim:

retention $z_i = x_i$, if $a_i < B$, or $d_i x_i$ if $a_i > B$

where: $d_i = \frac{B}{a_i}$

i.e. ceded loss, $W_i = 0$, if $a_i < B$, or $(1 - d_i)x_i$ if $a_i > B$.

$\therefore$ total ceded $W = \sum_{a_i > B} \left(1 - \frac{B}{a_i}\right)x_i$

The liability ceded, up to a fixed maximum, is usually divided in fixed proportions among several reinsurers ("first surplus" reinsurers), and the amount exceeding this maximum may be divided among other reinsurers ("second" and subsequent surplus reinsurers).

Net reinsurance premiums (less commission)

$$\Pi^* = \sum_{a_i > B} \left(1 - \frac{B}{a_i}\right)(M + h\Lambda)$$

3.1.3 Excess Reinsurance

The ceding company retains liability for any single claim up to a fixed amount, T, and cedes any excess of the claim over T (usually up to a fixed maximum):

retention $Z_i = x_i$ if $x_i < T$, or T if $x_i > T$

$\therefore$ $W_i = 0$ if $x_i < T$, or $x_i - T$ if $x_i > T$

$\therefore$ total ceded, $W = \sum_{x_i > T} (x_i - T)$.

The reinsurance premium is a fixed proportion, C, of the original gross premium, *regardless of the nominal sum insured.*

i.e. $\Pi^* = c(M + \Lambda + E) = C\Pi$

where E = total expense loading received by cedent,

Π = gross premiums received by cedent.

Π^* is designed to cover the reinsurers average share of the "fair" premium, the expenses of administering the treaty, and the risk to the reinsurer. Thus, in order to obtain a fair value of C, some estimate of the average fair premium must be made. This will depend on the distribution of the sizes of nominal sums insured, and is usually based on past experience of the average incidence of claims $> T$ (the "Burning Cost") for the particular ceding company. Clearly, in order for the assessment of reinsurance premiums to be reasonable, the treaty must be limited to lines of business where the distribution of sums insured has small dispersion and does not vary too much from year to year.

Excess reinsurance is often used to reinsure the cedent's liability retained from other forms of reinsurance.

Another form of excess reinsurance is called "*Excess Loss Ratio*" or "*Stop-Loss*" reinsurance. The amount ceded is the excess of total claims during the treaty over a predetermined proportion of total gross premium income.

i.e. total retention $Z = X$ if $x < t\Pi$

$$\text{or } Z = t\Pi \text{ if } x > t\Pi$$

(Note: if the original premium loading is a fixed proportion of the fair premium, i.e. $\Pi = (1 + c)M$, then the retention limit can be written as a proportion of the fair premium, $t'\Pi$, where $t' = t(1 + c)$.)

3.2 Reciprocal and Pool Reinsurance, Coinsurance

Reciprocal reinsurance is an arrangement between two or more primary insurers, whereby each company, r , retains a function $z_r(x_{r1}, x_{r2}, \ldots, x_{rn})$ of its own liabilities, and cedes $W_r = X_r - Z_r$ in exchange for a function $W_{rs}(x_{s1}, x_{s2}, \ldots, x_{sn})$ of the liabilities of each of the other participants, s , sometimes together with a net cash transfer. Clearly $W_r = \sum_u W_{ur}$.

The functions W may take several forms, and, commonly, selected policies are transferred in toto from company to company.

Reciprocal reinsurance may take the form of a treaty, under which all new policies are automatically issued jointly and divided in fixed proportions among the participating companies. This is called "*coinsurance*".

Another kind of reciprocal treaty is "*Pool Insurance*" under which each participant automatically effects an *excess* reinsurance on new policies with fixed retention, T , say, and where the other participants take a fixed proportion of the excess liability and the premiums.

i.e. retention of company r on i^{th} policy

$$= x_{ri} \quad \text{if} \quad x_{ri} < T$$

$$\text{or } T \quad \text{if} \quad x_{ri} > T$$

and W_{sr} = liability ceded to company s

$$= \sum_{x_{ri} > T} \{K_s x_{ri} - T\} \quad \text{where } K_s \text{ is a constant.}$$

3.4 Facultative Reinsurance

This type of contract applies to *individual policies* which are completely specified, and is commonly used to reinsure policies which are unusually large or hazardous, or in some other way exceptional.

The amounts ceded may follow the pattern of quota, surplus, or excess agreements, or they may take more complicated forms. The premiums will be those which the reinsurer considers necessary to cover the risks, and this will be a matter for negotiation.

Facultative reinsurance may be used on certain policies as a supplement to existing treaty reinsurance, so long as the treaty reinsurers agree to this.

Expenses for arranging facultative reinsurance tend to be larger than for treaties, but this type of contract allows greater flexibility.

4. RELATIVE MERITS OF DIFFERENT TYPES OF REINSURANCE CONTRACT; OPTIMALITY

It is shown by Lippman [16] that for a given amount of ceded expected liability (fair reinsurance premium) an excess reinsurance contract provides a higher utility (i.e. lower risk) to any cedent with a concave utility function than either surplus or quota reinsurance. This result is shown for a contract with nominal sums insured drawn randomly from any distribution, and could therefore be applied to treaty reinsurance.

It has also been shown by Borch [5] that, under certain conditions, a stop-loss contract gives the greatest possible reduction in variance for a given ceded expected liability; this has been shown to be true under more general conditions by Kahn [15] and Olin [18].

While these results are of great theoretical interest, they cannot be used as a general rule for determining the optimal kind of reinsurance required in practice. In particular, the following considerations should be taken into account.

i) The reinsurance company itself will have some aversion to risk, and it has been shown by Vajda [24] that a stop-loss contract gives the *greatest* possible to risk to the reinsurance company (in terms of variance). This makes it likely that the reinsurance company would require a larger risk loading for such a contract. In fact, even if the reinsurer's coefficient of risk aversion is small, the force of competition will tend to result in a higher price for this kind of cover.

ii) There is considerable variation in the cost of administering different types of contracts.

iii) There is normally an upper limit to the ceded amount on stop-loss

contracts. In general, different types of contracts were originally designed to meet specific needs. The stop-loss contract was designed to protect the cedent against a large number of moderate losses, and it may be that under such circumstances a loss exceeding the upper limit to the loss ceded is very improbable. These terms might not be so attractive to a cedent with a few very high-risk contracts.

In fact, it may turn out that for administrative and other reasons it is best to reinsure different sectors of an insurance company's portfolio under different schemes. Under these circumstances the limitations to the use of contingency reserves for different sectors would have to be considered, and the quantitative aspects of optimization would be very complicated.

However, there is no reason for not carrying out a thorough investigation of possible alternatives *with regard to prevailing costs*, for any portfolio. Facultative reinsurance is far easier to deal with, whereas for treaty reinsurance it is required to estimate the distribution of the volume and type of business to be transacted during the course of the treaty.

5. OPTIMIZATION UNDER DIFFERENT CRITERIA

5.1 Utility

In the ideal case in which a function $u(y)$, say, can be constructed so as to represent the utility of net assets y, the problem can be formulated as follows:

Expected utility of portfolio without reinsurance

$$= \int_0^\infty u[R + S - \xi]dF(\xi)$$

where R = reserves

S = premium reserve

ξ = total claims $\sim F$.

It is required to find a function, $Z(\xi_1, \xi_2, \ldots, \xi_n)$, the net retention, of individual claims $\xi_1, \xi_2, \ldots, \xi_n$, that will maximize the expected utility:

$$\int_0^\infty \cdots \int_0^\infty u[R + S - \Pi^* - Z(\underline{\xi})]dF_1(\xi_1) \ldots dF_n(\xi_n)$$

where Π^* = total reinsurance premium.

Usually Π^* will consist of the net "fair" premium, $M^* = E\{X - Z(\underline{\xi})\}$, together with a loading which, for example, may be proportional to the fair premium,

$$\text{i.e.} \quad \Pi^* = (1 + \lambda^*)M^* ,$$

or may be related to the dispersion of the risk $[X - Z(\underline{\xi})]$ to the reinsurer.[†] (See Benktender [2].)

Since the choice of available reinsurance contracts will in practice be limited, it will be necessary to consider separately certain classes of functions $Z(\underline{\xi})$; it may then be possible to optimize within each class and to compare the results.

For treaty reinsurance contracts it will also be necessary to have *some estimate of* the distribution of the numbers of different categories of policies which will be negotiated over the course of the treaty.

The optimization procedure is ullustrated with reference to a stop-loss treaty with no upper limit to the ceded liability. This type of contract, with fixed premium loading, λ^* , has been shown (see Arrow [1]) to give optimal utility to a risk-averse cedent for a given total risk X . Borch [13] illustrates how the optimal retention limit may be obtained. This can easily be extended to stop-loss *treaties*, where the retention limit is expressed as a certain fraction, τ , of the volume of business during the contract, M , which has known distribution, $H(M)$, say.

Assuming that, given M , X is distributed as $F(X \mid M)$, expected utility, given M ,

$$U(M,F) = \int_0^{M\tau} u[R + (1 + \lambda)M - \pi^*(M,\tau) - X]dF(X \mid M)$$

$$+ u[R + (1 + \lambda)M - \pi^*(M,\tau) - M\tau] \int_{M\tau}^{\infty} dF(X \mid M) \tag{1}$$

when λ = primary insurer's risk loading

[†]For *treaty* reinsurance Π^* is usually expressed as a percentage of the total premium income, Π , and so in proportional treaties the loading is a fraction of the primary insurer's loading.

and $\pi^*(M,\tau) = (1+\lambda^*) \int_{M\tau}^{\infty} (X - M\tau)dF(X \mid M)$ †. So $\frac{\partial \pi^*}{\partial \tau} = -M(1+\lambda^*)[1 - F(M\tau \mid M)]$. Letting $\bar{U}(\tau) = \int_0^{\infty} U(M,\tau)dH(M)$, we get

$$\frac{d\bar{U}(\tau)}{d\tau} = \int_0^{\infty} M[1 - F(M\tau \mid M)]\left\{(1+\lambda^*)\int_0^{M\tau} u'[R + (1+\lambda)M - \pi^* - X]dF(X \mid M)\right.$$

$$\left. -(1 - (1+\lambda^*)[1 - F(M\tau \mid M)])u'[R + (1+\lambda)M - \pi^* - M\tau\right\} dH(M) \tag{2}$$

For optimal τ we set expression (1) to 0; in particular, if M is fixed or known precisely, we set the expression in { } to 0. The method is illustrated for a particular U, F and H in the appendix.

For utility functions with risk aversion > 0 it is found that optimal τ increases with increasing λ and with decreasing $E[M]$.

5.2 Variance

When it is not possible to define a utility function, the criterion of variance of retained loss is often used for optimizing reinsurance policies. Some assessment must be made of the trade-off between reduction in variance and the cost of reinsurance. One possible approach which is applicable to facultative reinsurance of individual policies within an existing portfolio (Seal [23]) is to consider the maximum reduction in variance that can be obtained for a fixed reinsurance loading. The amount of net premium ceded is not restricted since this is assumed equal to the expected value of the ceded liability.

Assume that for a set of n independent policies, with variance σ_i^2, $i = 1,2, \ldots, n$, the reinsurer charges a premium loading $K_i\theta_i^*$ for

† If $dF(X \mid M)$ is a function of $X \mid M$, then $\pi^* \propto M$, which is in accordance with the method of reinsurance premium calculation commonly used.

insuring a proportion K_i of the individual claims. θ_i^* will be in general a function of the distribution of the claim, ξ_i. The question as to what proportion of each claim should be ceded subject to a given amount, C, of risk loading is solved as follows:

$$\underset{\underline{\alpha}}{\text{Minimize}} \quad \sum_{i=1}^{n} \sigma_i^2 (1 - \alpha_i)^2$$

$$\text{Subject to} \quad \sum_{i=1}^{n} \alpha_i \theta_i^* = C$$

i.e.
$$\underset{\underline{\alpha};L}{\text{Minimize}} \quad \sum_{i=1}^{n} \sigma_i^2 (1 - \alpha_i)^2 - L\left(C - \sum_{i=1}^{n} \alpha_i \theta_i^*\right) \tag{3}$$

where L is the Lagrange multiplier. This gives $\alpha_i = 1 - \dfrac{L\theta_i^*}{2\theta_i^2}$ where L can be obtained by resubstitution in (3). If the reinsurance company charged premiums consistent with those suggested by a quadratic utility function θ_i^* would be approximately proportional to $\rho\sigma_i^2$ where ρ is its coefficient of risk aversion.

In this case $1 - \alpha_i \propto \rho L$, constant for all i, which is equivalent to prorata quota reinsurance.

In other cases the above formulae may give some $\alpha_i < 0$, in which case adjustments need to be made; in the particular case where $\theta_i^* \propto \dfrac{\sigma_i^2}{a_i}$ (a_i = nominal sum insured), the optimal contract in this class will be surplus reinsurance.

The variance-minimizing approach can be extended, in a similar way as for the utility approach, to treaty reinsurance, given detailed enough information about the distribution of policies negotiated during the course of the treaty.

5.3 Reciprocal Reinsurance

Reciprocal reinsurance is an important special case, for which the criteria of utility and variance are both commonly used.

An exchange of liabilities (and premiums) between insurance companies can increase the utility for both of them. For example, in the simplest case, if two companies have identical independent portfolios, by simply sharing each claim equally they both reduce the variance and (assuming they are both risk averse) increase the utility of their portfolios.

The more general case is examined by Borch [3]. Suppose Company A has risk distribution $F_A(X_A)$ and Company B has risk distribution $F_B(X_B)$. And if

A retains risk Z_A and cedes to B $W_A = X_A - Z_A$

and

B retains risk Z_B and cedes to A $W_B = X_B - Z_B$

then

$$Var(X_A) = Var(W_A) + Var(Z_A) + 2Cov(W_A, Z_A) \qquad (4)$$

and

$$Var(X_B) = Var(W_B) + Var(Z_B) + 2Cov(W_B, Z_B) \qquad (5)$$

Let A's total liability be $Q_A = Z_A + W_B + (\pi_A^* - \pi_B^*)$ where $(\pi_A^* - \pi_B^*)$ is the net transfer of premiums.

Then $Var(Q_A) = Var(Z_A) + Var(W_B)$ and from (4) and (5):

$$Var(Q_A) = Var(X_A) - Var(W_A) + Var(W_B) - 2Cov(W_A, Z_A)$$

and a similar expression for B's liability.

It is clear that under the criterion of minimizing variance, for a given variance of net ceded liability, the higher the correlation between ceded and retained liabilities, the better. In this sense, a pro rata exchange

of liabilities, for which the correlation between W and Z is 1, will be optimal.

Using utility functions U_A and U_B , for $Q_A = Q$ Company A's expected utility will be:

$$U_A(Q) = \iint U_A(R_A + S_A - Q)dF_A(X_A)dF_B(X_B)$$

where R_A and S_A are the free and premium reserves and $U_B(Q) = \iint U_B(R_A + S_A + Q - X_A - X_B)dF_A(X_A)dF_B(X_B)$.

The "Pareto optimal set" is defined as the set of Q such that there is not $\bar{Q}$ which is better for *both* companies, i.e. for $Q \in$ Pareto optimal set,

$$\exists \text{ no } \bar{Q} \text{ s.t. } \begin{cases} U_A(\bar{Q}) > U_A(Q) \\ \text{and } U_B(\bar{Q}) > U_B(Q) \end{cases}.$$

Borch [4] shows that a necessary and sufficient condition for Q to belong to this set is:

$$U'_B(R_B + S_B + Q - X_A - X_B) = K \cdot U'_A(R_A + S_A - Q)$$

for some positive K .

The actual value of K , and hence the required Q , will be a matter for negotiation. Nash [17] suggests using the value which maximizes:

$$[U_A(Q) - U_A(0)] \cdot [U_B(Q) - U_B(0)]$$

where $U_A(0)$ and $U_B(0)$ are the expected utilities without reinsurance.

i.e. $\qquad U(0) = \int U(R + S - X)dF(X) \quad$ for A and B .

These concepts may be extended (see Borch [7]) to several insurance companies cooperating or competing in the reinsurance market.

5.4 Probability of Ruin

The probability of ruin of an insurance company depends on the relationship between the size and number of contracts in its portfolio, their risk distributions, and the size of the reserves. The following very simple example illustrates this:

Suppose a portfolio contains n identical independent policies with claims distributed normally, $N(\mu, \sigma^2)$.

Suppose the company charges premiums $p = \mu(1 + \lambda)$. Then total premiums $P = n\mu(1 + \lambda)$ and total claims $X \sim N(u\mu, n\sigma^2)$.

If initial reserves are R , probability of ruin $= \Phi\left(\frac{R + n\mu}{\sigma\sqrt{n}}\right)$

This takes a maximum value $\Phi\left(\frac{2\sqrt{R\lambda\mu}}{\sigma}\right)$ when $n = \frac{R}{\lambda\mu}$. Clearly the probability decreases with R and increases with σ^2 . For $n \ll \frac{R}{\lambda\mu}$, the size of the total risk will not *threaten the reserves*, and for $n \gg \frac{R}{\lambda\mu}$, the accumulated loading will be sufficient to cover the risk.

Benktender [2] considers an insurance company with normally distributed risk, so that probability of ruin $= \Phi\left(\frac{R + L}{\Sigma}\right)$ when Σ^2 = variance of whole portfolio. If a new independent risk with premium $\pi = \mu + \theta$ and variance σ^2 is introduced, the probability or ruin becomes $\Phi\left(\frac{R + L + \theta}{\sqrt{\Sigma^2 + \sigma^2}}\right)$.

If the probability of ruin is *not to be increased*, this requires

$$\theta \geq (R + L)\left\{\left(1 + \frac{\sigma^2}{\Sigma^2}\right)^{1/2} - 1\right\} \simeq \frac{R + L}{2\Sigma^2}\cdot\sigma^2 \quad \text{if} \quad \sigma^2 \ll \Sigma^2 . \qquad (6)$$

This suggests that under this criterion the company should charge a loading proportional to the *variance* of the risk. Alternatively, if the loading normally charged is proportional to the standard deviation of the

risk, i.e. $\theta = \kappa\sigma$, (6) gives a maximum value to the standard deviation of any risk that can be accepted, and hence the maximum retention of the risk is to be reinsured:

$$\sigma \leq \kappa\sqrt{\frac{(R + L)}{2\sum^2}} .$$

Note, however, that under condition (6), the larger the premium reserves, and the smaller the variance of the existing portfolio, the larger will be the loading required for a given risk. If *reinsurance* companies were to use this criterion, the larger companies would tend to be more conservative. That it often appears to be so, may reflect the fact that insurance companies have to pay more for this greater margin of safety.

The theory of random walks may be applied to the probability of ruin, and this subject is extensively covered by Seal [23]. If a dividend is automatically paid out when net assets reach a certain fixed level, the probability of ultimate ruin becomes 1. The question of optimality under these conditions is discussed by Borch [12].

5.5 Value of Dividends

The discounted value of future dividends is sometimes used as a criterion of optimality. This will be more suitable for companies conducting business of a short-term nature than, for example, life insurance companies, who will tend to give greater weight to continuing solvency.

A very general approach is given by Pechlivanides [20].† A "state of nature", $\underline{X}_t$, at stage t , is defined as the vector of claims at the end of the stage for the group of participating insurance and reinsurance companies which constitute the "market".

†The notation used here differs somewhat from that used previously.

For any state of nature $\underline{X}_t$, a "*price function*" $P(\underline{X}_t)$ is defined, when $P(\underline{X}_t)d\underline{X}_t$ represents the market's assessment of the value of a contract to pay \$1 if the state of nature falls in the range $(\underline{X}_t;\underline{X}_t + d\underline{X}_t)$. This takes into account the market's assessment of the probability of such an event.

The quantity π_t , defined as $\int_{\underline{X}_t} P(\underline{X}_t)d\underline{X}_t$, is equivalent to the value of a contract to pay \$1 in *any* event, and so $\pi_t = \frac{1}{1+i}$ where i is the prevailing rate of interest.

The market value of a particular company's wealth, y_t , (i.e. assets less liabilities) at the end of the stage is

$$\int_{\underline{X}_t} Y_t(\underline{X})P(\underline{X}_t)d\underline{X}_t \tag{7}$$

and the company can trade in these assets for a post-reinsurance wealth Z_t , which is a function of $\underline{X}_t$, and hence of the vector of net assets, $\underline{Y}_t$, without reinsurance.

The value of the post-reinsurance wealth must be the same as the traded-in value, i.e.

$$\int_{\underline{X}_t} Y_t P(\underline{X}_t)d\underline{X}_t = \int_{\underline{X}_t} Z(\underline{X}_t)P(\underline{X}_t)P(\underline{X}_t)d\underline{X}_t \tag{8}$$

Each company strives to maximize its expected utility (according to its own assessment of the distribution of $\underline{X}_t$) subject to (7) above. Pechlivanides gives a set of conditions for the existence of a "market equilibrium" (Pareto optimal set), subject to the market conservation condition that the total redistributed wealth, $\sum_{\substack{\text{all}\\ \text{companies}}} Z(\underline{X}_t)$ is equal to $\sum_{\substack{\text{all}\\ \text{companies}}} Y_t$, the total wealth without reinsurance. (9)

In the simplest case (considered below) of a single cedent-reinsurer relationship, the "state of nature" is fully described by X_t, the liabilities of the cedent. The reinsurer is responsible for quoting a price function $P(X_t)$. The cedent can choose to pay any dividend C_t (payable immediately). The premium income, p_t, during stage t is assumed known.

Given total initial reserves R_t, the cedent can choose a post-reinsurance wealth, $R(X_t)$, subject to condition (8), i.e.

$$\int_{X_t} \left\{ \frac{R_t - C_t}{\pi_t} + p_t - X_t \; P(X_t) dX_t \right\} = \int_{X_t} R(X_t) P(X_t) dX_t \qquad (10)$$

The criterion of optimality is the utility of dividends over a given number of stages, represented by the discounted sum of $u(C_i)$, the utilities of future individual dividends, plus the discounted value of $u(R_0)$ where R_0 is the final wealth.†

The problem of optimal choice of C_t and $R(X_t)$ is solved by Dynamic Programming, and closed form solutions can be obtained for the "Linear Risk-Tolerance" Class of utility functions.

In practice, the premium income, p_t, would be a random variable, and the available choice of reinsurance contracts, $R(X_t)$, would be severely limited. Although it is unlikely that prices quoted as any reinsurance market would be entirely consistent with any price function $P(X)$, such a function could provide an excellent basis for a consistent method of premium assessment (Pechlivanides [20], p. 84 and 122).

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†By suitable choice of an alternative utility function for R_0, it might be possible to give appropriate weight to the long-term solvency of the company.

6. OPEN QUESTIONS

Of the methods of calculating optimal reinsurance considered, the most promising would appear to be the dynamic approach (Pechlivanides [19]), but from a practical point of view, the following factors would have to be considered:

i) How to give weight to the continuing solvency of the company apart from the value of dividends.

ii) How to take account of random variation in premium income.

It might also be useful to consider the cost of negotiating treaties, and to incorporate into the dynamic model possible choice in the *term* of a treaty. Decisions would depend on recent experience of premium income and claims distribution, and hence Bayesian methods could be useful; there would be a trade-off between saving on the cost of negotiation and lack of precision in the estimates of future experience over the course of the treaty.

The main disadvantage of the dynamic approach is that it depends on a totally flexible range of insurance contracts and a consistent and predictable method of pricing.

From this point of view, an assessment of utility on a year-to-year basis, taking into account current market prices, seems more applicable in practice. However, this requires that the utility function used should be such as to truly represent the value of net assets at the end of the period being considered with regard to the whole of the future, (and this depends on the solution to the dynamic problem). It might be possible to make some compromise between the two approaches.

Another difficulty would be that the insurance company may have several portfolios that have to be considered separately for the purposes of reinsurance, and there may be restrictions on the movement of reserves.

BIBLIOGRAPHY

[1] Arrow, K. J., "Optimal Insurance and Generalized Deductibles," Skandinavian Actuarial Journal, pp. 1-42, (1974).

[2] Benktander, G., "Some Aspects on Reinsurance Profits and Loadings," ASTIN, Vol. 5, Part 3, (1971).

[3] Borch, K. H., "Reciprocal Reinsural Treaties Seen as a Two-person Cooperative Game," S.A.J., No. 1-2, (1960a).

[4] ____________, "The Safety Loading of Reinsurance Premiums," S.A.J., No. 3-4, (1960b).

[5] ____________, "An Attempt to Determine the Optimum Amount of Stop Loss Reinsurance," Trans. of the 16th Inter. Cong. Act., Vol. 1, (1960c).

[6] Borch, K.H., "Reciprocal Reinsurance Treaties," ASTIN, Vol. 1, Part 4, (1960d).

[7] ____________, "A Contribution to the Theory of Reinsurance Markets," Skandinavian Actuarial Journal, (1962).

[8] ____________, "Control of a Portfolio of Insurance Contracts," ASTIN, Vol. 4, Part 1, (1966).

[9] ____________, "Dynamic Decision Problems in an Insurance Company," ASTIN, Vol. 5, Part 1, (1968a).

[10] ____________, "General Equilibrium in the Economics of Uncertainty," in RISK AND UNCERTAINTY, K. H. Borch and J. Mossin (eds.), MacMillan, (1968b).

[11] ____________, "Economic Equilibrium under Uncertainty," International Economic Review, Vol. 9, No. 3, (1968c).

[12] ____________, "The Theory of Risk," Journal of the Royal Statistical Society, Series B, Vol. 29, (1967).

[13] ____________, "Optimal Insurance Arrangements," ASTIN, Vol. 8, Part 3, (1975).

[14] Bühlmann, H., MATHEMATICAL METHODS IN RISK THEORY, Springer-Verlag, (1970).

[15] Kahn, P. M., "Some Remarks on a Recent Paper by Borch," ASTIN, Vol. 1, Part 5, (1961).

[16] Lippman, S., "Optimal Reinsurance," Western Management Science Institute, University of California, Los Angeles, (1971).

[17] Nash, J. F., "The Bargaining Problem," Econometrica, Vol. 18, (1950).

[18] Ohlin, J., "A Generalization of a Result by Borch and Kahn on the Optimal Properties of Stop Loss Reinsurance," ASTIN, Vol. 5, (1969).

[19] Pechlivanides, P., "Approximations by Orthogonal Functions in Casualty Insurance," ORC 73-24, Operations Research Center, University of California, Berkeley, (1973).

[20] ________________, "Reinsurance Market Mechanisms and Dividend Strategies for an Insurance Company," ORC 75-17, Operations Research Center, University of California, Berkeley, (1975).

[21] Pratt, J. W., "Risk Aversion in the Small and in the Large," Econometrica, Vol. 32, No. 1-2, (1964).

[22] Reinarz, R. C., "Property and Liability Reinsurance Management," Mission, Fullerton, California, (1969).

[23] Seal, H. L., STOCHASTIC THEORY OF A RISK BUSINESS, Wiley and Sons, Inc., (1969).

[24] Vajda, S., "Minimum Variance Reinsurance," ASTIN, Vol. 2, Part 2, (1962).

APPENDIX

Example of method of calculation of optimal retention for stop-loss reinsurance.

With reference to Section 5.1, suppose we make the following assumptions:

1. Utility function is $u(x) = A(1 - e^{-\mu x})$.

   This has constant risk aversion, $-\frac{u''(x)}{u'(x)} = \mu$.

   As $x \to \infty$, $u(x) \to A$, its maximum value.

   Since we want to relate this to initial wealth, R , we define $\phi = \frac{u(R)}{A}$

$$\text{i.e., } \mu = 1/R\{\log [1/(1 - \phi)]\} = \psi/R \text{ , say} \tag{1}$$

$$\text{Then } u(x) = A\left(1 - e^{-\frac{x}{R}\psi}\right) \tag{2}$$

and the exponent is now dimensionless, regarding ψ as the parameter of the utility function.

2. Claims distribution $F(X \mid M)$ is Γ (2,M)

$$\text{i.e. } dF(X \mid M) = \left(\frac{2}{M}\right)^2 \cdot Xe^{-2x/M}dX$$

(and $E(X \mid M) = M$, as required).

   If we write $Y = \frac{2x}{M}$, and $(Y \mid M) \sim G(Y \mid M)$

$$\text{then } dG(Y \mid M) = Ye^{-Y}dY \text{ , and } E(Y \mid M) = 2 \tag{3}$$

independent of M†.

3. The retention is τM , a fraction of the volume.
4. M has some known distribution $M \sim H(M)$.

   Then, from 5.2, writing $Y = \frac{2x}{M}$

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† See note on p. 13.

$$\frac{d\bar{U}(\tau)}{d\tau} = \int_0^\infty M[1 - G(2\tau)]\left\{(1 + \lambda^*)\int_0^{2\tau} u'\left[R + (1 + \lambda)M - \pi^* - \frac{MY}{2}\right]dG(Y)\right.$$

$$\left. - (1 - (1 + \lambda^*)[1 - G(2\tau)])u'[R + (1 + \lambda)M - \pi^* - M\tau]\right\}dH(M)$$

$$\text{where} \quad \pi^* = (1 + \lambda^*)\int_{2\tau}^\infty \left(\frac{MY}{2} - M\tau\right)Ye^{-Y}dY = a(\tau)M \text{ , say,}$$

$$\text{where} \quad a(\tau) = (1 + \lambda^*)(1 + \tau)e^{-2\tau}$$

$$\therefore \frac{d\bar{U}(\tau)}{d\tau} = [1 - G(2\tau)]\left\{(1 + \lambda^*)\int_0^\infty\int_0^{2\tau} \frac{AM\psi}{R}\exp[-\psi/R(R + M[1 + \lambda - a - Y/2])]dG(Y)dH(M)\right.$$

$$\left. -(1 - (1 + \lambda^*)[1 - G(2\tau)])\int_0^\infty \frac{AM\psi}{R}\exp[-\psi/R(R + M[1 + \lambda - a - \tau])]dH(M)\right\} \tag{4}$$

$$= A\psi e^{\psi}\ [1 - G(2\tau)]\left\{(1 + \lambda^*)\int_0^{2\tau} Ye^{-Y}\int_0^\infty J\exp[-\psi J(1 + \lambda - a - Y/2)]dB(J)dY\right.$$

$$\left. -(1 - (1 + \lambda^*)[1 - G(2\tau)])\int_0^\infty J\exp[-\psi J(1 + \lambda - a - \tau)]dB(J)\right\}$$

$$\text{where} \quad J = M/R \sim B(J) \quad \text{given} \quad R \text{ .}$$

If M , and hence J , are of known constant value, J_0 (as in facultative reinsurance, for example), by setting $\frac{d\bar{U}(\tau)}{d\tau}$ to 0 , and cancelling common factors, we obtain

$$(1 + \lambda^*)\int_0^{2\tau} Y\exp\left[-Y\left(1 - \frac{\psi J_0}{2}\right)\right]dY = (1 - (1 + \lambda^*)(1 + 2\tau)e^{-2\tau})\exp(\psi J_0\tau) \tag{5}$$

for optimal τ .

Writing $\rho = 1 - \frac{\psi J_0}{2} = 1 - \frac{\psi M}{2R}$

$$(1 + \lambda^*)\left\{-\frac{2\tau}{\rho}\exp[-2\tau\rho] + \frac{1}{\rho 2}(1 - \exp[-2\tau\rho])\right\}$$

$$= (1 - (1 + \lambda^*)(1 + 2\tau)e^{-2\tau})\exp[2\tau(1 - \rho)] \qquad (6)$$

$$\therefore e^{2\tau\rho} = 1 + 2\tau\rho + \rho^2\left\{\frac{e^{2\tau}}{1 + \lambda^*} - (1 + 2\tau)\right\}$$

and this can easily be solved numerically for τ . The solution gives τ increasing with λ^* (i.e. the higher the reinsurance loading, the greater the retention), and also τ increasing with ρ (i.e. the larger M , the volume, and the larger μ , the coefficient of risk aversion, the lower the optimal retention).

If J is now allowed to vary, e.g. suppose J has gamma distribution with mean J_0 ,

$$dB(J) = \left(\frac{K}{J_0}\right)^K \frac{J^{K-1}}{(K - 1)!} e^{-\frac{K}{J_0} \cdot J} \, dJ ,$$

where K is a measure of the precision of variable J , then from Equation 5.1, substituting $J = \frac{M}{R}$ and taking expectations w.r.t. J , we have:

$$\bar{U}(\tau) \propto \int_0^\infty \left\{ \int_0^{M\tau} \left[\exp\left(-\frac{J_0}{K} J\right) - \exp\left(-\psi - \frac{J}{J_0}[K + \psi J_0(1 + \lambda - a(\tau) - Y/2)]\right)\right] dG(Y)\right.$$

$$\left. + \left[\exp\left(-\frac{J_0}{K} J\right) - \exp\left(-\psi - \frac{J}{J_0}[K + \psi J_0(1 + \lambda - a(\tau) - \tau)]\right)\right][1 - G(2\tau)]\right\} J^{K-1} dJ$$

and this is clearly finite and differentiable for

$$\tau \in T = \{\tau \mid [K + \psi J_0(1 + \lambda - a(\tau) - \tau)] > 0\} ,$$

$$\text{and} \quad \bar{U}(\tau) \to -\infty \quad \text{as} \quad [K + \psi J_0(1 + \lambda - a(\tau) - \tau)] \to 0^+ ,$$

and is infinitely negative for $\tau \notin T$.

If T is not empty, since $\bar{U}(\tau)$ is bounded above, and since $\left.\frac{d\bar{U}(\tau)}{d\tau}\right|_{\tau=0} > 0$ (see below), then there is at least one solution to $\frac{d\bar{U}(\tau)}{d\tau} = 0$, giving a local maximum.

If we consider cases where $\{\tau = 0\} \in T$, i.e. $[K + \psi J_0(\lambda - \lambda^*)] > 0$, it can be shown that there is a unique τ_0 S.t. $[K + \psi J_0(1 + \lambda - a(\tau) - \tau)] = 0$, and we look for a solution to $\frac{d\bar{U}(\tau)}{d\tau} = 0$ in $(0,\tau_0)$. From Equation (4)

$$\frac{d\bar{U}(\tau)}{d\tau} \propto (1 + \lambda^*) \int_0^{2\tau} Ye^{-Y} \int_0^{\infty} J^K \exp\left(- \frac{J}{J_0}\left[K + \psi J_0\left(1 + \lambda - a(\tau) - \frac{Y}{2}\right)\right]\right) dJdY$$

$$-[1 -(1 + \lambda^*)(1 + 2\tau)e^{-2\tau}] \int_0^{\infty} J^K \exp\left(- \frac{J}{J_0}[K + \psi J_0(1 + \lambda - a(\tau) - \tau)]\right) dJ$$

and this is > 0 for $\tau = 0$

$\therefore$ If $\frac{d\bar{U}(\tau)}{d\tau} = 0$,

$$(1 + \lambda^*) \int_0^{2\tau} Ye^{-Y} \frac{dY}{[K + \psi J_0(1 + \lambda - a(\tau) - Y/2)]^K}$$

$$-[1 - (1 + \lambda^*)(1 + 2\tau)e^{-2\tau}] \cdot \frac{1}{[K + \psi J_0(1 + \lambda - a(\tau) - \tau)]^K} = 0$$

i.e.,

$$(1 + \lambda^*) \int_0^{2\tau} Ye^{-Y} \left\{ \frac{[K + \psi J_0(1 + \lambda - a(\tau) - \tau)]^K}{[K + \psi J_0(1 + \lambda - a(\tau) - Y/2)]^K} \right\}$$

$$- [1 - (1 + \lambda^*)(1 + 2\tau)e^{-2\tau}] = 0$$

and this has a solution with $\dfrac{d^2\bar{U}(\tau)}{d\tau^2} < 0$.

It can easily be shown that the optimal value, $\hat{\tau}$, say, increases with λ and decreases with J_0 . Also, $\hat{\tau}$ decreases with K (the precision) for fixed J_0 in a neighborhood of any K for which $1 + \lambda - a(\hat{\tau}) - \hat{\tau} > 0$. We note also that as $K \to \infty$, i.e., as J tends to become more precisely fixed at J_0 , the expression { } $\to$ $\exp(-\psi J_0[Y/2 - \tau])$ and we get back to Equation (5).